

$F(R)$ gravity's rainbow and its Einstein counterpart

S. H. Hendi^{1,2}, B. Eslam Panah¹, S. Panahiyan^{1,3}, and M. Momennia¹

¹ *Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran*

² *Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Maragha, Iran*

³ *Physics Department, Shahid Beheshti University, Tehran 19839, Iran*

Motivated by UV completion of general relativity with a modification of a geometry at high energy scale, it is expected to have an energy dependent geometry. In this paper, we introduce charged black hole solutions with power Maxwell invariant source in the context of gravity's rainbow. In addition, we investigate two classes of $F(R)$ gravity's rainbow solutions. At first, we study energy dependent $F(R)$ gravity without energy momentum tensor, and then we obtain $F(R)$ gravity's rainbow in the presence of conformally invariant Maxwell source. We study geometrical properties of the mentioned solutions and compare their results. We also give some related comments regarding to thermodynamical behavior of the obtained solutions and discuss thermal stability of the solutions.

I. INTRODUCTION

An accelerated expansion of the Universe was confirmed by various observational evidences. The luminosity distance of Supernovae type Ia [1, 2], the anisotropy of cosmic microwave background radiation [3], and also wide surveys on galaxies [4] confirm such accelerated expansion. On the other hand, baryon oscillations [5], large scale structure formation [6], and weak lensing [7] also propose such an accelerated expansion of the Universe.

After discovery of such an expansion in 1998, understanding its theoretical reasons presents one of the fundamental open questions in physics. Identifying the cause of this late time acceleration is a challenging problem in cosmology. Physicists are interested in considering this accelerated expansion in a gravitational background and they proposed some candidates to explain it. For example, a positive cosmological constant leads to an accelerated expansion, but it is plagued by the fine tuning problem [8–12]. In other words, the left hand side of Einstein equations can modify by the cosmological constant as a geometrical modification or it can be interpreted as a kinematic term on the right hand side with the equation of state parameter $w = -1$. By considering $w < -1/3$ for a source term, it is possible to further modify this approach. This consideration has interpretation of Dark Energy which has been investigated in literature [13–19]. In dark energy models, the acceleration expansion of the universe is due to an unknown ingredient added to the cosmic pie. The effects of this unknown ingredient is extracted by modifying the stress energy tensor of the Einstein equation with a matter which is different from than the usual matter and radiation components.

On the other hand, it is proposed that the presence of accelerated expansion of the universe indicate that the standard general relativity requires modification. To do so, one can generalize the Einstein field equations to obtain a modified version of gravity. There are different branches of modified gravity with various motivations, such as brane world cosmology [20–22], Lovelock gravity [23–27], scalar-tensor theories [28–35], and $F(R)$ gravity [36–51].

Modifying general relativity opens a new way to a large class of alternative theories of gravity ranging from higher dimensional physics [52–54] to non-minimally coupled (scalar) fields [55–58]. Regarding various models of modified gravity, we will be interested in $F(R)$ gravity [59–65] based on replacing the scalar curvature R with a generic analytic function $F(R)$ in the Hilbert-Einstein action. Some viable functional forms of $F(R)$ gravity may be reconstructed starting from data and physically motivated issues.

However, the field equations of $F(R)$ gravity are complicated fourth order differential equations, and it is not easy to find exact solutions. In addition, adding a matter field to $F(R)$ gravity makes the field equations much more complicated. On the other hand, regarding constant curvature scalar model as a subclass of general $F(R)$ gravity can simplify the field equations. Also, one can extract exact solutions of $F(R)$ gravity coupled to a traceless energy momentum tensor with constant curvature scalar [66]. For example, considering exact solutions of $F(R)$ gravity with conformally invariant Maxwell (CIM) field as a matter source has been investigated [67, 68].

General relativity coupled to a nonlinear electrodynamics attracts significant attentions because of its specific properties in gauge/gravity coupling. Interesting properties of various models of the nonlinear electrodynamics have been investigated before [69–81]. Power Maxwell invariant (PMI) theory is one of the interesting branches of the nonlinear electrodynamics which its Lagrangian is an arbitrary power of Maxwell Lagrangian [82–84]. The PMI theory is more interesting with regard to Maxwell field, and for unit power, it reduces to linear Maxwell theory. This nonlinear electrodynamics enjoys conformal invariance when the power of Maxwell invariant is a quarter of spacetime dimensions, and this is one of the attractive properties of this theory. In other words, one can obtain traceless energy-momentum tensor for a special case "power = dimensions/4" which leads to conformal invariance. It is notable to mention that this idea has been considered to take advantage of the conformal symmetry to construct the analogues of the four dimensional Reissner-Nordström solutions with an inverse square electric field in arbitrary dimensions [85].

From the gravitational point of view, it is possible to show that the electric charge and cosmological constant can be extracted, simultaneously, from pure $F(R)$ gravity (without matter field: $T_{\mu\nu} = 0$) [51]. In this paper, we are going to obtain d -dimensional charged black hole solutions from gravity's rainbow, pure $F(R)$ gravity's rainbow as well as $F(R)$ gravity's rainbow with CIM source and compare them to obtain their direct relation.

In order to build up special relativity from Galilean theory, one has to take into account an upper limit for velocity of particles. The same method could be used to restrict particles from obtaining energies no more than specific energy, the so-called Planck energy scale. This upper bound of energy may modify dispersion relation which is known as double special relativity [86]. Generalization of this doubly special relativity to incorporate curvature leads to gravity's rainbow [87]. In gravity's rainbow, spacetime is a combination of the temporal and spatial coordinates as well as energy functions. The existence of such energy functions indicates that, the particle probing the spacetime can acquire specific range of energies which in essence leads to formation of a rainbow of energy.

There are several features for gravity's rainbow which highlight the importance of such generalization. Among them one can point modification in energy-momentum dispersion relation which is supported by studies that are conducted in string theory [88], loop quantum gravity [89] and experimental observation [90]. Also, existence of remnant for black holes [91] which is proposed to be a candidate for solving the information paradox [92]. In addition, this theory admits the usual uncertainty principle [93, 94].

Recently, there has been a growing interest in energy dependent spacetimes [95–100]. Different classes of black holes have been investigated in the context of gravity's rainbow [101–104]. The hydrostatic equilibrium equation of stars in the presence of gravity's rainbow has been obtained [105]. Furthermore, a study regarding the gravity's rainbow and compact stars has been done [106]. In Ref. [107], the effects of gravity's rainbow for wormholes have been investigated. Moreover, the influences of gravity's rainbow on gravitational force have been investigated [108]. Also, Starobinsky model of $F(R)$ theory in gravity's rainbow has been studied in Ref. [109]. In addition, gravity's rainbow has interesting effects on the early universe [110–112].

The main motivations for studying black holes in the presence of gravity's rainbow given as follows. First of all, due to the high energy properties of the black holes, it is necessary to consider quantum corrections of classical perspectives. One of the methods to include quantum corrections of gravitational fields is by considering an energy dependent spacetime. In fact, it was shown that the quantum correction of gravitational systems could be observed in dependency of spacetime on the energy of particles probing it which is gravity's rainbow point of view [93, 113, 114]. Since we are modifying our point of view to an energy dependent spacetime, it is expected to find its effects on the properties of black holes, especially in the context of their thermodynamics. This is another motivation for considering gravity's rainbow generalization. Also, there are specific achievements for gravity's rainbow in the context of black holes which among them one can name: modified uncertainty principle [93, 94], existence of remnants for the black holes [91, 102], furnishing a bridge towards Horava—Lifshitz gravity [115], providing possible solution toward information paradox [91] and finally being UV completion of Einstein gravity [116]. In addition, as it was pointed out before, in the context of cosmology, it presents a possible solution toward big bang singularity problem [110–112]. On the other hand, $F(R)$ gravity provides correction toward gravitational sector of the Einstein theory of gravity. The importance of this correction is highlighted in the context of black holes. In order to have better picture regarding the physical nature of the black holes, one may consider high energy regime effects, as well (considering that the concepts of Hawking radiation was derived by studying black holes in semi classical/quantum regime). Here, we apply $F(R)$ gravity's rainbow to find the effects of $F(R)$ generalization as well as energy dependency of spacetime on the black hole solutions.

The outline of our paper is as follows. In Section II, we are going to investigate black hole solutions in Einstein-gravity's rainbow with PMI and CIM fields. Then, we want to investigate conserved and thermodynamic quantities of the solutions and check the first law of thermodynamics. In Section III, we will obtain black hole solutions of $F(R)$ gravity's rainbow with CIM source and check the first law of thermodynamics. We also discuss thermal stability of these solutions and criteria governing stability/instability in Sec. IV. Then, we consider pure $F(R)$ gravity's rainbow and compare these solutions with $F(R)$ gravity's rainbow with CIM source and give some related comments regarding to its thermodynamical behavior. Finally, we finish our paper by some conclusions.

II. EINSTEIN GRAVITY'S RAINBOW IN THE PRESENCE PMI FIELD

Here, we are going to introduce d -dimensional solutions of the Einstein-gravity's rainbow in the presence of PMI field with the following Lagrangian

$$\mathcal{L} = R - 2\Lambda + (\kappa\mathcal{F})^s, \quad (1)$$

where R and Λ are, respectively, the Ricci scalar and the cosmological constant. In Eq. (1), the Maxwell invariant is $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field and A_μ is the gauge potential. As we

mentioned before, in the limit $s = 1$ with $\kappa = -1$, the Lagrangian (1) reduces to the Lagrangian of Einstein-Maxwell gravity. Since the Maxwell invariant is negative, henceforth we set $\kappa = -1$, without loss of generality. Using the variation principle, we can find the field equations the same as those obtained in Ref. [117].

A. Black hole Solutions

Here, we will obtain charged rainbow black hole solutions with negative cosmological constant in d -dimensions. It is notable that the charged rainbow black hole solutions in Einstein gravity coupled to nonlinear electromagnetic fields have been studied in Ref. [118]. In this paper, we want to extend the spacetime to d -dimensions and obtain black hole solutions in the presence of PMI field as a matter source. The rainbow metric for spherical symmetric spacetime in d -dimensions can be written as

$$ds^2 = -\frac{\psi(r)}{f(E)^2} dt^2 + \frac{1}{g(E)^2} \left[\frac{dr^2}{\psi(r)} + r^2 d\Omega^2 \right], \quad (2)$$

where

$$d\Omega^2 = d\theta_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2. \quad (3)$$

Using the metric (2) with the field equations, one can obtain the following solutions for the metric function as well as gauge potential

$$\psi(r) = 1 - \frac{m}{r^{d-3}} - \frac{2\Lambda r^2}{(d-1)(d-2)g(E)^2} + \begin{cases} \frac{-2^{\frac{d-1}{2}} g(E)^{d-3} (qf(E))^{d-1}}{r^{d-3}} \ln\left(\frac{r}{l}\right), & s = \frac{d-1}{2} \\ \frac{r^{2(2s-1)^2} \left(\frac{2[qf(E)g(E)(2s-d+1)]^2 r^{\frac{-2d+4}{2s-1}}}{(2s-1)^2} \right)^s}{[(d-1)(d-2)-2(d-2)s]g(E)^2}, & \text{otherwise} \end{cases}, \quad (4)$$

$$A_\mu = h(r)\delta_\mu^t, \quad (5)$$

where the consistent $h(r)$ function is $-q \ln \frac{r}{l}$ or $-qr^{\frac{2s-d+1}{2s-1}}$ for $s = \frac{d-1}{2}$ and $s \neq \frac{d-1}{2}$, respectively. In addition, m and q are integration constants which are, respectively, related to the mass and electric charge of the black hole. We should also mention that we consider $s > 1/2$ for obtaining well-behaved electromagnetic field. It is notable that by replacing $s = 1$ and $g(E) = f(E) = 1$, the solutions (4) reduce to the following higher dimensional Reissner-Nordström black hole solutions

$$\psi(r) = 1 - \frac{m}{r^{d-3}} - \frac{2\Lambda r^2}{(d-1)(d-2)} + \frac{2(d-3)q^2}{(d-2)r^{2(d-3)}}. \quad (6)$$

In order to investigate the geometrical structure of these solutions, we first look for the essential singularity(ies). The Ricci scalar can be written as

$$R = \frac{2d}{d-2}\Lambda + \begin{cases} \frac{2^{\frac{d-1}{2}} (qf(E)g(E))^{d-1}}{r^{d-1}}, & s = \frac{d-1}{2} \\ \frac{4s-d}{d-2} \left(\frac{2[qf(E)g(E)(2s-d+1)]^2 r^{\frac{-2d+4}{2s-1}}}{(2s-1)^2} \right)^s, & \text{otherwise} \end{cases}, \quad (7)$$

and the behavior of Kretschmann scalar is

$$\begin{aligned} \lim_{r \rightarrow 0} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} &\longrightarrow \infty, \\ \lim_{r \rightarrow \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} &= \frac{8d}{(d-1)(d-2)^2} \Lambda^2, \end{aligned} \quad (8)$$

therefore, confirm that there is a curvature singularity at $r = 0$. In addition, the Kretschmann scalar is $\frac{8d}{(d-1)(d-2)^2} \Lambda^2$ for $r \rightarrow \infty$, which confirms that the asymptotical behavior of the charged rainbow black hole is adS. It is worthwhile to mention that the asymptotical behavior of these solutions is independent of the rainbow functions and the power

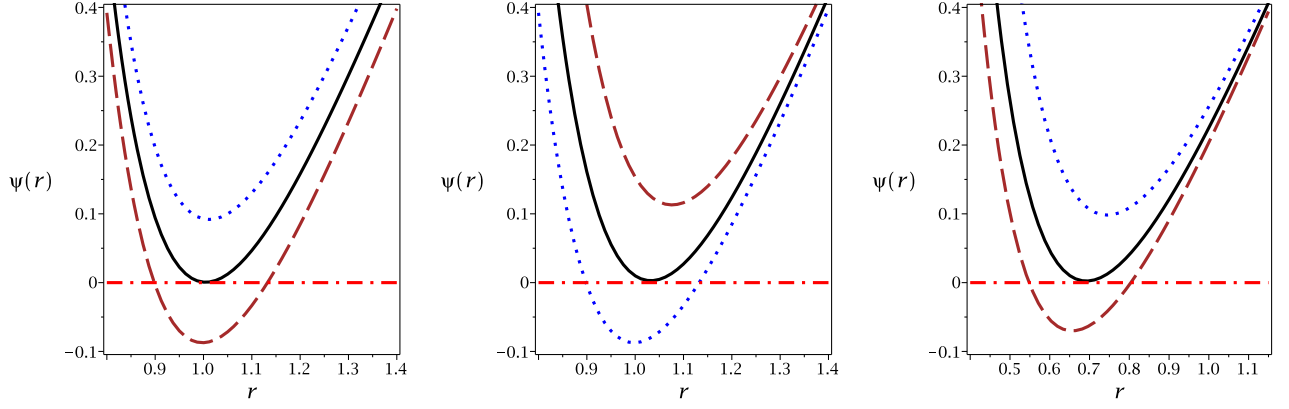


FIG. 1: $\psi(r)$ versus r for $g(E) = 1.1$, $f(E) = 1.1$, $\Lambda = -1$ and $d = 4$.

Left diagram: for $s = 0.7$, $q = 1$, $m = 1.95$ (dotted line), $m = 1.87$ (continuous line) and $m = 1.80$ (dashed line).

Middle diagram: for $s = 0.7$, $m = 2$, $q = 0.5$ (dotted line), $q = 0.7$ (continuous line) and $q = 1.1$ (dashed line).

Right diagram: for $m = 2$, $q = 1$, $s = 1.125$ (dotted line), $s = 1.108$ (continuous line) and $s = 1.095$ (dashed line).

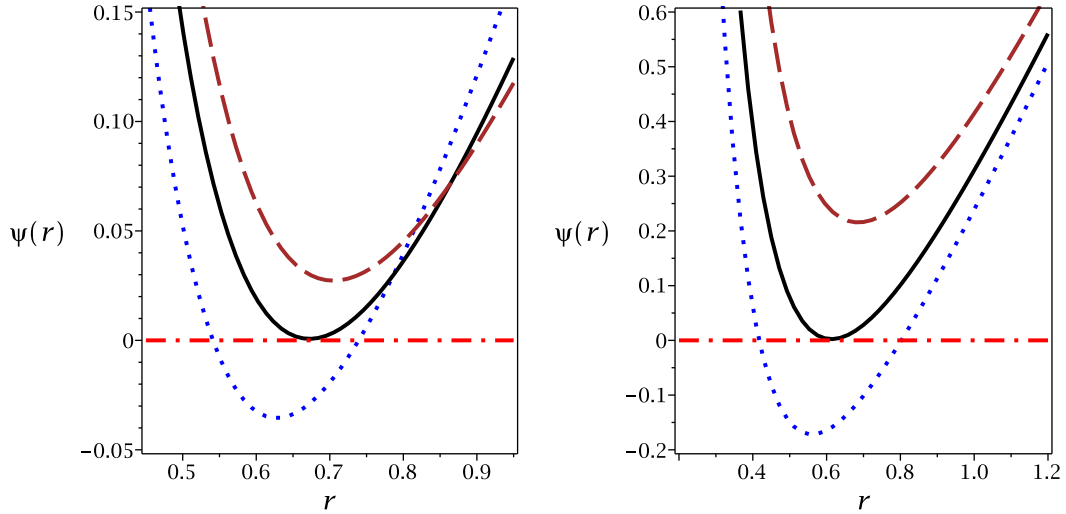


FIG. 2: $\psi(r)$ versus r for $m = 2$, $q = 0.8$, $\Lambda = -1$, and $d = 4$.

Left diagram: for $f(E) = 1.00$, $s = 1.06$, $g(E) = 1.40$ (dotted line), $g(E) = 0.86$ (continuous line) and $g(E) = 0.68$ (dashed line).

Right diagram: for $g(E) = 1.0$, $s = 1.1$, $f(E) = 0.95$ (dotted line), $f(E) = 1.02$ (continuous line) and $f(E) = 1.10$ (dashed line).

of PMI source. This independency comes from the fact that rainbow functions are sensible in high energy regime such as near horizon and one expects to ignore its effects far from the black hole (we only consider high energy regime). In addition, at large distances, the electric field of Maxwell and PMI theories vanishes, and therefore, one may expect to ignore the effects of electric charge far from the origin. In order to investigate the possibility of the horizon, we plot the metric function versus r in Figs. 1 and 2. It is evident that depending on the choices of values for different parameters, we may encounter with two horizons (inner and outer horizons), one extreme horizon and without horizon (naked singularity).

Special Case $s = \frac{d}{4}$:

Here, we are going to investigate the special case $s = \frac{d}{4}$, the so-called conformally invariant Maxwell field (for more details, see [68, 85]). It was shown that for $s = \frac{d}{4}$, the energy-momentum tensor will be traceless and the corresponding electric field will be proportional to r^{-2} in arbitrary dimensions, as it takes place for the Maxwell field

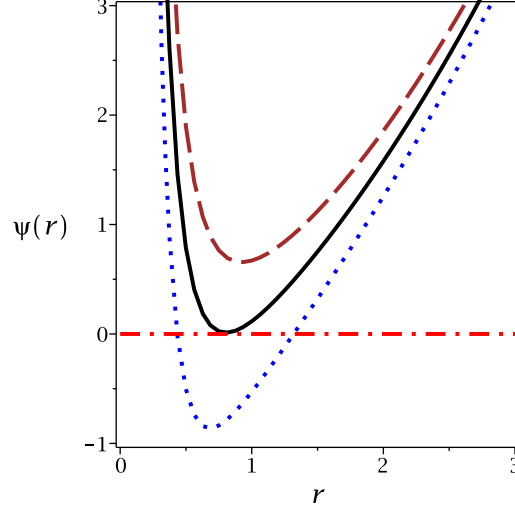


FIG. 3: $\psi(r)$ versus r for $q = 1$, $d = 4$, $f(E) = 1.1$, $\Lambda = -1$, $g(E) = 1.0$, $m = 3.00$ (dotted line), $m = 2.46$ (continuous line) and $m = 1.80$ (dashed line).

in 4-dimensions. Therefore, we consider $s = \frac{d}{4}$ into Eqs. (4) and (5) to obtain

$$\psi(r) = 1 - \frac{m}{r^{d-3}} - \frac{2\Lambda r^2}{(d-1)(d-2)g(E)^2} + \frac{2^{\frac{d-4}{4}}(qf(E)g(E))^{\frac{d}{2}}}{g(E)^2 r^{d-2}}, \quad (9)$$

$$h(r) = -\frac{q}{r}. \quad (10)$$

Considering $g(E) = f(E) = 1$ in the above equation, we obtain the higher dimensional Reissner-Nordström black hole solution. In order to make more investigations regarding the properties of these solutions, we plot the metric function (9) in Fig. 3. It is evident that the behavior of Einstein-CIM-rainbow black holes (Eq. (9)) is similar to Einstein-PMI-rainbow black holes (see Figs. 1 and 2).

B. Thermodynamics

Here, we are going to calculate the conserved and thermodynamic quantities and check the first law of thermodynamics for such black hole. In order to calculate thermodynamic quantities, we start with temperature. Using the definition of surface gravity, one can calculate the Hawking temperature of the black hole as

$$T_+ = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu)}, \quad (11)$$

where $\chi = \partial/\partial t$ is the Killing vector. So, the Hawking temperature for the rainbow black holes in the presence of PMI source can be written as

$$T_+ = -\frac{1}{2\pi f(E)} \left\{ \frac{\Lambda r_+}{(d-2)g(E)} - \frac{(d-3)g(E)}{2r_+} \right\} - \mathcal{T}, \quad (12)$$

where \mathcal{T} is

$$\mathcal{T} = \begin{cases} \frac{2^{(d-5)/2} [qf(E)g(E)]^d}{\pi q f(E)^2 g(E)^2 r_+^{d-2}} & s = \frac{d-1}{2} \\ \frac{2^{s-2} (2s-1) r_+}{(d-2) \pi f(E) g(E)} \left[\frac{qf(E)g(E)(d-2s-1)}{(2s-1) r_+^{(d-2)/(2s-1)}} \right]^{2s} & otherwise \end{cases}, \quad (13)$$

in which r_+ satisfies $f(r = r_+) = 0$. Moreover, the electric potential U is defined by [119, 120]

$$U = A_\mu \chi^\mu|_{r \rightarrow reference} - A_\mu \chi^\mu|_{r=r_+} \quad (14)$$

So for these black holes, we obtain

$$U = \begin{cases} q \ln\left(\frac{r_+}{l}\right) & s = \frac{d-1}{2} \\ q r_+^{\frac{(2s-3)}{(2s-1)}} & otherwise \end{cases}. \quad (15)$$

The entropy of the black holes satisfies the so-called area law of entropy in Einstein gravity. It means that the black hole's entropy equals to one-quarter of horizon area [121–123]. Therefore, the entropy of the black holes in d -dimensions is

$$S = \frac{1}{4} \left(\frac{r_+}{g(E)} \right)^{d-2}. \quad (16)$$

In order to obtain the electric charge of the black holes, one can calculate the flux of the electromagnetic field at infinity, so we have

$$Q = \begin{cases} -\frac{1}{\pi} 2^{\frac{(d-3)}{2}} (d-1) [qf(E)]^{d-2} & s = \frac{d-1}{2} \\ \frac{s(2s-1)}{8(2s-d+1)\pi q f(E)g(E)^{d-1}} \left[\frac{\sqrt{2}(2s-d+1)qf(E)g(E)}{(2s-1)} \right]^{2s} & otherwise \end{cases}. \quad (17)$$

The spacetime introduced in Eq. (2), have boundaries with timelike ($\xi = \partial/\partial t$) Killing vector field. It is straightforward to show that the total finite mass can be written as

$$M = \frac{(d-2)m}{16\pi f(E)g(E)^{d-3}}. \quad (18)$$

Now, we are in a position to check the validity of the first law of thermodynamics. To do so, one can employ the following relation

$$dM(S, Q) = \left(\frac{\partial M(S, Q)}{\partial S} \right)_Q dS + \left(\frac{\partial M(S, Q)}{\partial Q} \right)_S dQ. \quad (19)$$

It is a matter of calculation to show that following equalities hold

$$T = \left(\frac{\partial M}{\partial S} \right)_Q, \quad U = \left(\frac{\partial M}{\partial Q} \right)_S, \quad (20)$$

which confirm that the first law of thermodynamics is valid for the obtained thermodynamic and conserved quantities. We will address thermodynamic stability in section 4.

III. $F(R)$ GRAVITY'S RAINBOW IN THE PRESENCE OF CIM FIELD

Here, we consider $F(R) = R + f(R)$ gravity's rainbow with the CIM field as a matter source, which leads to a traceless energy-momentum tensor. For d -dimensions, the equations of motion for the $F(R)$ gravity's rainbow with CIM source are

$$R_{\mu\nu} (1 + f_R) - \frac{g_{\mu\nu}}{2} F(R) + (g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) f_R = 8\pi T_{\mu\nu}, \quad (21)$$

$$\partial_\mu \left(\sqrt{-g} F^{\mu\nu} \mathcal{F}^{\left(\frac{d}{4}-1\right)} \right) = 0, \quad (22)$$

in which $f_R = \frac{df(R)}{dR}$. In order to extract black hole solutions in $F(R)$ gravity's rainbow coupled to the matter field, it is essential that we consider a traceless energy-momentum tensor.

A. Black hole Solutions

We want to obtain the black hole solutions for constant scalar curvature ($R = R_0 = \text{const}$). Using Eqs. (5) and (22) with metric (2), one can find

$$h(r) = \frac{-\mathcal{B}}{r}, \quad \& \quad F_{tr} = \frac{\mathcal{B}}{r^2}, \quad (23)$$

where \mathcal{B} is an integration constant.

Regarding the trace of Eq. (21), one finds $R = R_0 = \frac{df(R_0)}{2(1+f_R)-d} \equiv 4\Lambda$. Substituting the mentioned R_0 into Eq. (21), we obtain the following equation

$$R_{\mu\nu} (1 + f_R) - \frac{1}{d} g_{\mu\nu} R_0 (1 + f_R) = 8\pi T_{\mu\nu}. \quad (24)$$

Now, considering the metric (2) with Eq. (24), one can write the field equations in the following forms

$$\frac{g(E)^2}{d-2} r \psi''(r) + g(E)^2 \psi'(r) + \frac{2}{d(d-2)} r R_0 = \frac{2^{\frac{d}{4}} [\mathcal{B} g(E) f(E)]^{\frac{d}{2}}}{4(1+f_R) r^{d-1}}, \quad (25)$$

$$r g(E)^2 \psi'(r) + (d-3) g(E)^2 [\psi(r) - 1] + \frac{r R_0}{d} = -\frac{2^{\frac{d-4}{4}} [\mathcal{B} g(E) f(E)]^{\frac{d}{2}}}{(1+f_R) r^{d-2}}, \quad (26)$$

which are corresponding to tt (or rr) and $\varphi\varphi$ (or $\theta\theta$) components, respectively. After some calculations, we can obtain the metric function in the following form

$$\psi(r) = 1 - \frac{m}{r^{d-3}} + \frac{2^{\frac{d-4}{4}} (\mathcal{B} f(E) g(E))^{\frac{d}{2}}}{g(E)^2 (1+f_R) r^{d-2}} - \frac{R_0 r^2}{d(d-1) g(E)^2}. \quad (27)$$

In order to have well-behaved solutions, hereafter, we restrict ourselves to $f'(R_0) \neq -1$. We are going to study the general structure of the solutions. For this purpose, we must investigate the behavior of the Kretschmann scalar. The Kretschmann scalar goes to infinity (∞) at the origin ($\lim_{r \rightarrow 0} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \rightarrow \infty$). Also the Kretschmann scalar is finite at infinity ($\lim_{r \rightarrow \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \rightarrow \frac{2}{d(d-1)} R_0^2$). So, it shows that the Kretschmann scalar diverges at $r = 0$, and is finite for $r \neq 0$. Therefore, there is a curvature singularity located at $r = 0$. When we define $R_0 = \frac{2d}{d-2} \Lambda$, the spacetime will be asymptotically adS. In order to have physical solutions of this gravity, we should restrict ourselves to $f_R \neq -1$. Also, for $f_R > -1$, when we substitute $R_0 = \frac{2d}{d-2} \Lambda$ and $\mathcal{B} = q[1+f_R]^{\frac{2}{d}}$ in Eq. (27), this solution turns into the black hole solution of Einstein-CIM-gravity's rainbow (9) with the same behavior. In other words, the solutions obtained for $F(R)$ gravity's rainbow in the presence of CIM source (Eq. (27)) are similar to the Einstein-CIM-rainbow solutions. Therefore, these black hole solutions have at least one horizon.

B. Thermodynamics

Here, considering $F(R)$ gravity's rainbow in the presence of CIM source, we want to obtain the Hawking temperature for the black hole solutions. Using the definition of surface gravity and Eq. (27), one can calculate the Hawking temperature as

$$T_+ = \frac{(1+f_R) [d(d-3)g(E)^2 - R_0 r_+^2] r_+^{d-2} - d \{2^{d-4} [\mathcal{B} f(E) g(E)]^{2d}\}^{1/4}}{4d\pi f(E) g(E) r_+^{d-1} (1+f_R)}. \quad (28)$$

For this theory, the electric charge and the electric potential of the conformally invariant $F(R)$ rainbow black hole are, respectively,

$$Q = \frac{d2^{d/4-1}}{16\pi} \left(\frac{\mathcal{B} f(E)}{g(E)} \right)^{d/2-1}, \quad (29)$$

$$U = \frac{\mathcal{B}}{r_+}. \quad (30)$$

In the context of modified gravity theories the area law may be generalized and one can use the Wald entropy associated with the Noether charge [124]. In order to obtain the entropy of black holes in $F(R) = R + f(R)$ theory, one can use a modification of the area law [125]

$$S = \frac{1}{4} \left(\frac{r_+}{g(E)} \right)^{d-2} (1 + f_R), \quad (31)$$

which reveals that the area law does not hold for the obtained black hole solutions in $F(R)$ gravity's rainbow.

In addition, using the Noether approach, we find that the finite mass can be obtained as

$$M = \frac{(d-2)(1+f_R)m}{16\pi f(E)g(E)^{d-3}}. \quad (32)$$

After calculating the conserved and thermodynamic quantities, we can check the validity of the first law of black hole thermodynamics. Although $F(R)$ gravity and rainbow functions may modify various quantities, it is straightforward to show that the modified conserved and thermodynamic quantities satisfy the first law of thermodynamics as $dM = TdS + UdQ$. The modified conserved and thermodynamic quantities suggest a deep connection between the horizon thermodynamics and geometrical properties in modified gravity.

IV. THERMODYNAMIC STABILITY OF CIM, PMI AND F(R) MODELS

In this section, we employ the canonical ensemble approach toward thermal stability and phase transition of the solutions. The canonical ensemble is based on studying the behavior of heat capacity. In this approach, roots of the heat capacity which are exactly same as the roots of temperature are denoted as bound points which separate non-physical solutions (having negative temperature) from physical ones (positive temperature). On the other hand, the divergencies of the heat capacity are denoted as second order phase transition points. In other words, in divergence point of the heat capacity, system goes under a second order phase transition.

The system is in thermal stable state if the signature of heat capacity is positive. In case of the negative heat capacity, system may acquire stability by going under phase transition or it may always be unstable which is known as non-physical case. The heat capacity is obtained as

$$C_Q = \frac{T}{\left(\frac{\partial^2 M}{\partial S^2} \right)_Q} = \frac{T}{\left(\frac{\partial T}{\partial S} \right)_Q}. \quad (33)$$

Now, considering Eqs. (12), (16), (28) and (31), one can find following heat capacities

$$C_Q = \begin{cases} -\frac{(d-2)r_+^{d-2}}{4g^{d-2}(E)} \left(\frac{\left[(d-3)g^2(E) - \frac{2\Lambda r_+^2}{d-2} \right] r_+^{d-3} - [\sqrt{2}qf(E)g(E)]^{d-1}}{\left[(d-3)g^2(E) + \frac{2\Lambda r_+^2}{d-2} \right] r_+^{d-3} - (d-2)[\sqrt{2}qf(E)g(E)]^{d-1}} \right), & CIM \\ -\frac{(d-2)r_+^{d-2}}{4g^{d-2}(E)} \left(\frac{(d-2)(d-3)g^2(E) - \Theta(2s-1)r_+^2 - 2\Lambda r_+^2}{(d-2)(d-3)g^2(E) - \Theta(2s[d-3]+1)r_+^2 + 2\Lambda r_+^2} \right), & PMI \\ -\frac{(d-2)(1+f_R)r_+^{d-2}}{4g^{d-2}(E)} \left(\frac{(1+f_R) \left[(d-3)g^2(E) - \frac{2\Lambda r_+^2}{d-2} \right] r_+^{d-2} - \sqrt[4]{2^{d-4}(qf(E)g(E))^2 d}}{(1+f_R) \left[(d-3)g^2(E) + \frac{2\Lambda r_+^2}{d-2} \right] r_+^{d-2} - \frac{(d-1)}{(d-2)} \sqrt[4]{2^{d-4}(qf(E)g(E))^2 d}} \right), & F(R) \end{cases} \quad (34)$$

in which

$$\Theta = \left(\frac{2 [qf(E)g(E)(2s-d+1)]^2 r_+^{\frac{4-2d}{2s-1}}}{(2s-1)^2} \right)^s. \quad (35)$$

Due to complexity of the obtained solutions, it is not possible to find the divergence and bound points analytically, and therefore, we employ numerical approach. We present the results of our study through different diagrams (see Figs. 4-10).

Evidently, for CIM and PMI, the thermodynamical stability and phase transitions are highly sensitive to variations of the energy functions. As one can see, for these two cases by increasing energy functions, the stability conditions

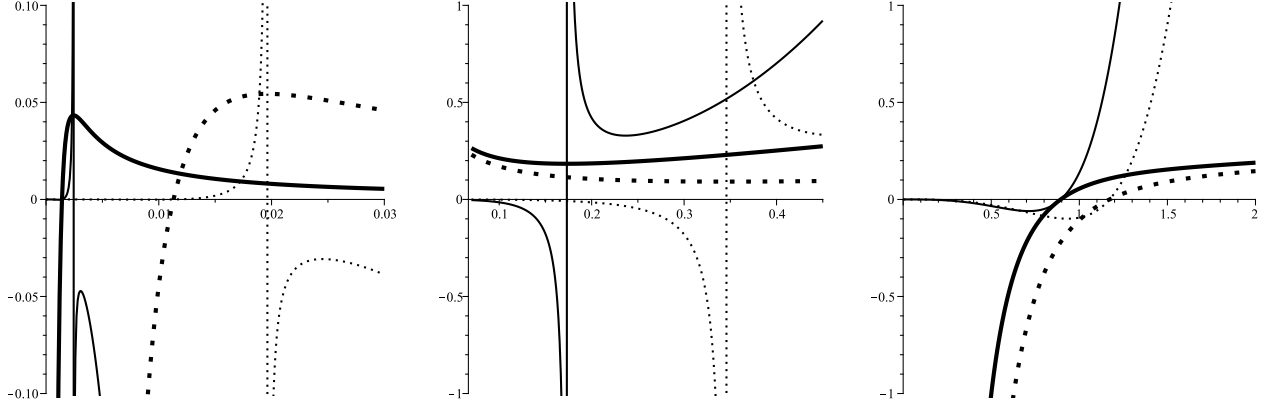


FIG. 4: For different scales (CIM case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$ and $d = 5$; left and middle panels: $g(E) = f(E) = 0.1$ (continues line) and $g(E) = f(E) = 0.2$ (dotted line). right panel: $g(E) = f(E) = 0.9$ (continues line) and $g(E) = f(E) = 1$ (dotted line).

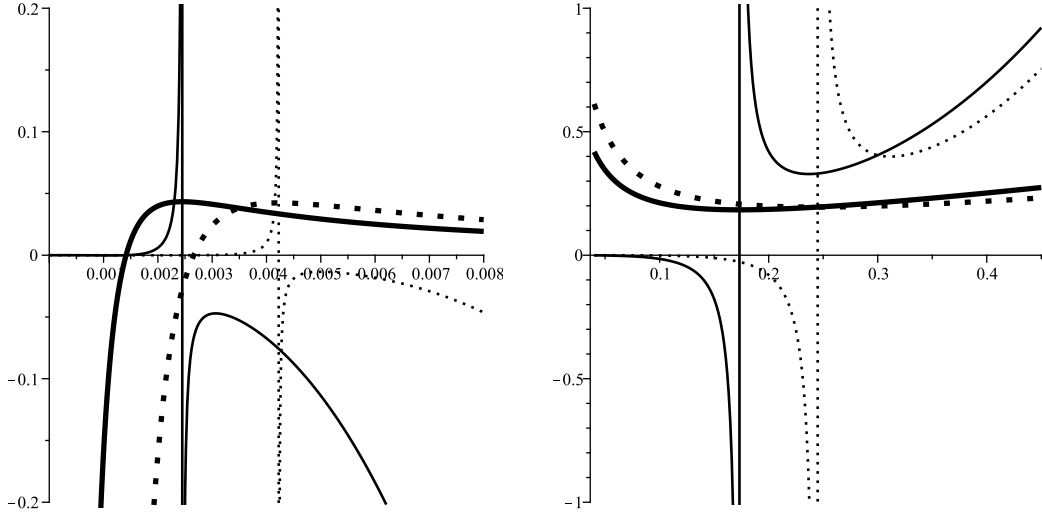


FIG. 5: For different scales (CIM case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$ and $g(E) = f(E) = 0.1$; $d = 5$ (continues line) and $d = 6$ (dotted line).

will be modified completely. For small values of energy functions, two divergencies and one bound point are observed (left and middle panels of Fig. 4 and left panel of Fig. 8). The stable solutions exist between bound point and smaller divergency and also after larger divergency. Physical but unstable solutions are between two phase transition points and otherwise they are non-physical. The divergencies are marking second order phase transitions. By increasing energy functions, these behaviors will be modified into two single states of non-physical unstable and physical stable solutions (right panel of Fig. 4 and left panel of Fig. 8). Interestingly, such effects are not observed for $F(R)$ gravity. In this case, the variations of energy functions act as a translation factor. In other words, they shift the places of bound and divergence points (right panel of Fig. 9).

As for the PMI solutions, increasing PMI parameter, s , leads to modification in thermal structure of the solutions. For small values of this parameter, two regions of non-physical unstable and physical stable states exist (left panel of Fig. 6). By increasing s , a bound point and two singularities for the heat capacity are formed. The bound point and smaller divergency are decreasing functions of s while larger divergency is an increasing function of it (see Figs. 6 and 7). Similar behavior is observed for f_R in $F(R)$ gravity (left and middle panels of Fig. 9). As for the effects of dimensionality, in these three theories, the bound and divergence points are increasing functions of this parameter (see Fig. 5, right panel of Fig. 8 and Fig. 10), the only exception is an interesting behavior for smaller divergence point for $F(R)$, where an abnormal behavior is observed (see middle panel of Fig. 10).

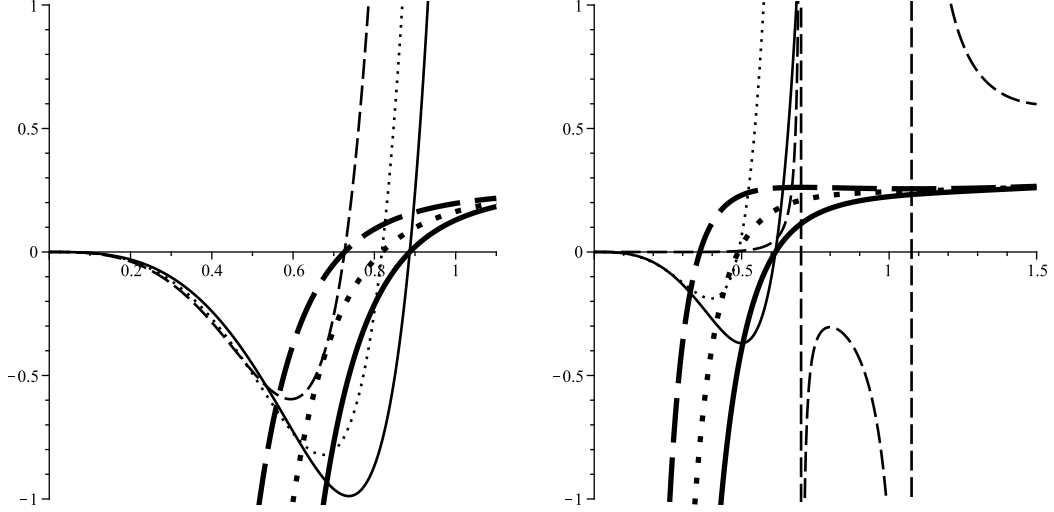


FIG. 6: For different scales (PMI case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$, $g(E) = f(E) = 0.7$ and $d = 5$; left panel: $s = 0.9$ (continues line), $s = 1$ (dotted line) and $s = 1.1$ (dashed line). right panel: $s = 1.2$ (continues line), $s = 1.3$ (dotted line) and $s = 1.4$ (dashed line).

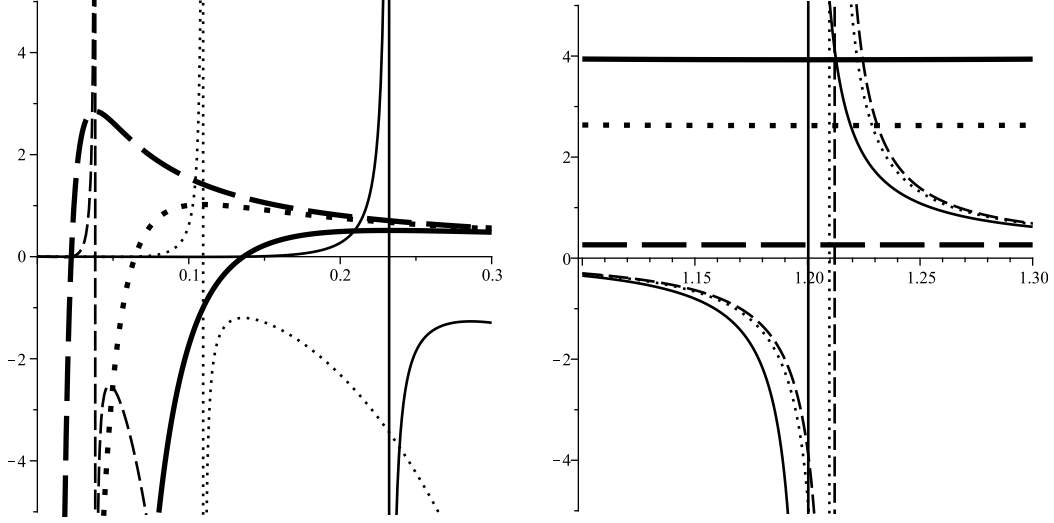


FIG. 7: For different scales (PMI case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$, $g(E) = f(E) = 0.7$ and $d = 5$; $s = 1.6$ (continues line), $s = 1.7$ (dotted line) and $s = 1.8$ (dashed line).

V. PURE $F(R)$ GRAVITY'S RAINBOW

In this section, we are going to obtain the charged solutions of pure $F(R)$ gravity's rainbow. Therefore, we consider $F(R) = R + f(R)$ gravity without matter field ($T_{\mu\nu}^{matt} = 0$) for the spherical symmetric spacetime in d -dimensions.

Using the field equation (21) and metric (2), one can obtain the following independent sourceless equations

$$F_R'' + \frac{\mathcal{J}_1}{2r\psi(r)}F_R' - \frac{\mathcal{J}_2}{2r\psi(r)}F_R = \frac{F(R)}{2\psi(r)g(E)^2}, \quad (36)$$

$$\mathcal{J}_1 F_R' - \mathcal{J}_2 F_R = \frac{rF(R)}{g(E)^2}, \quad (37)$$

$$F_R'' + \frac{\mathcal{J}_3}{r\psi(r)}F_R' - \frac{(\mathcal{J}_3 - d + 3)}{r^2\psi(r)}F_R = \frac{F(R)}{2\psi(r)g(E)^2}, \quad (38)$$

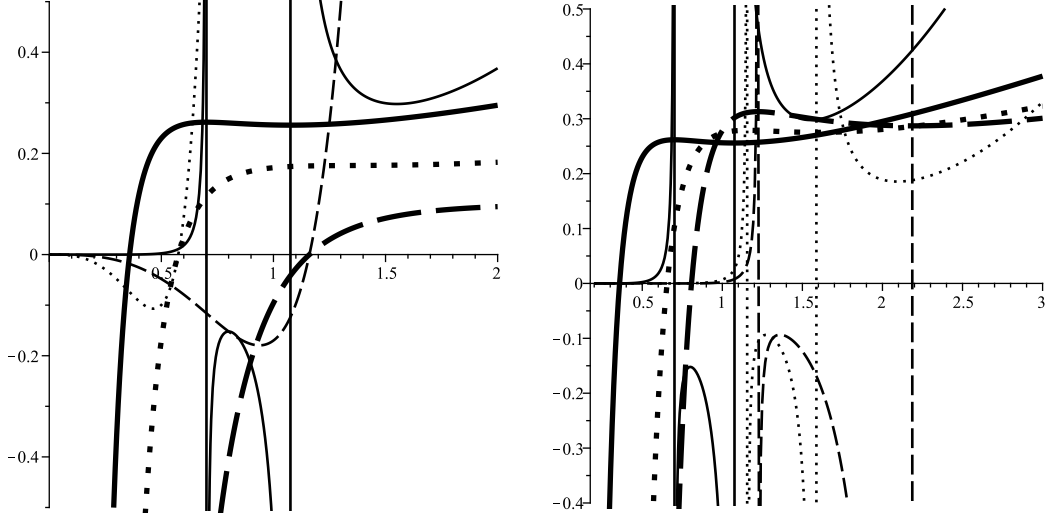


FIG. 8: For different scales (PMI case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$ and $s = 1.4$; left panel: $d = 5$, $g(E) = f(E) = 0.7$ (continues line), $g(E) = f(E) = 1$ (dotted line) and $g(E) = f(E) = 1.7$ (dashed line). right panel: $g(E) = f(E) = 0.7$, $d = 5$ (continues line) and $d = 7$ (dotted line).

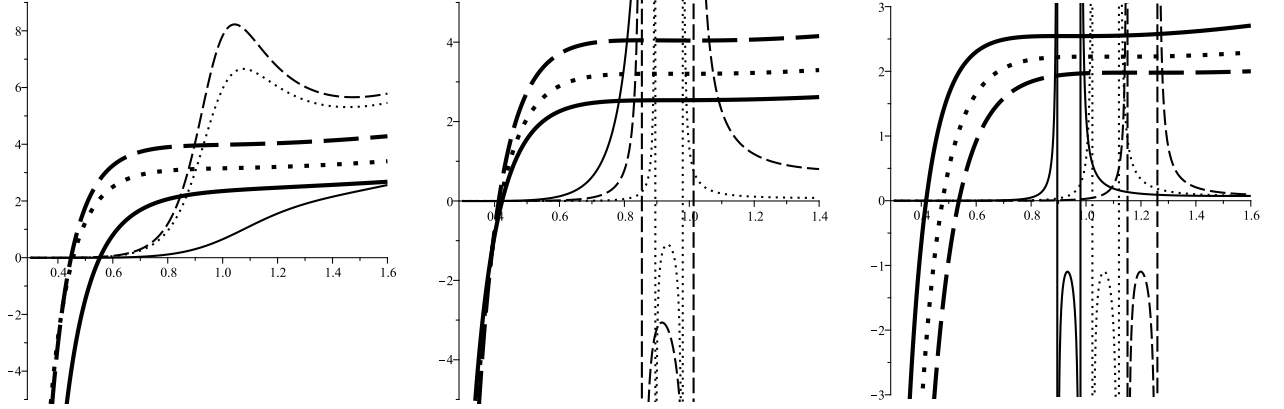


FIG. 9: For different scales ($F(R)$ case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$ and $d = 5$; left panel: $g(E) = f(E) = 0.7$, $f_R = 0$ (continues line), $f_R = 0.9$ (dotted line) and $f_R = 1$ (dashed line). middle panel: $g(E) = f(E) = 0.7$, $f_R = 1.4$ (continues line), $f_R = 1.5$ (dotted line) and $f_R = 1.6$ (dashed line). right panel: $f_R = 1.5$, $g(E) = f(E) = 0.7$ (continues line), $g(E) = f(E) = 0.8$ (dotted line) and $g(E) = f(E) = 0.9$ (dashed line).

where these equations are, respectively, corresponding to tt , rr and $\varphi\varphi$ components of gravitational field equation. It is notable that, $F_R = \frac{dF(R)}{dR}$, prime and double prime denote the first and second derivative with respect to r , and

$$\begin{aligned}\mathcal{J}_1 &= r\psi'(r) + 2(d-2)\psi(r), \\ \mathcal{J}_2 &= r\psi''(r) + (d-2)\psi'(r), \\ \mathcal{J}_3 &= r\psi'(r) + (d-3)\psi(r).\end{aligned}$$

A. Black hole Solutions

Here, we study black hole solutions in pure $F(R)$ gravity's rainbow with constant Ricci scalar ($F_R'' = F_R' = 0$), so it is easy to show that Eqs. (36)-(38) reduce to the following forms

$$g(E)^2 [r\psi''(r) + (d-2)\psi'(r)] F_R = -rF(R), \quad (39)$$

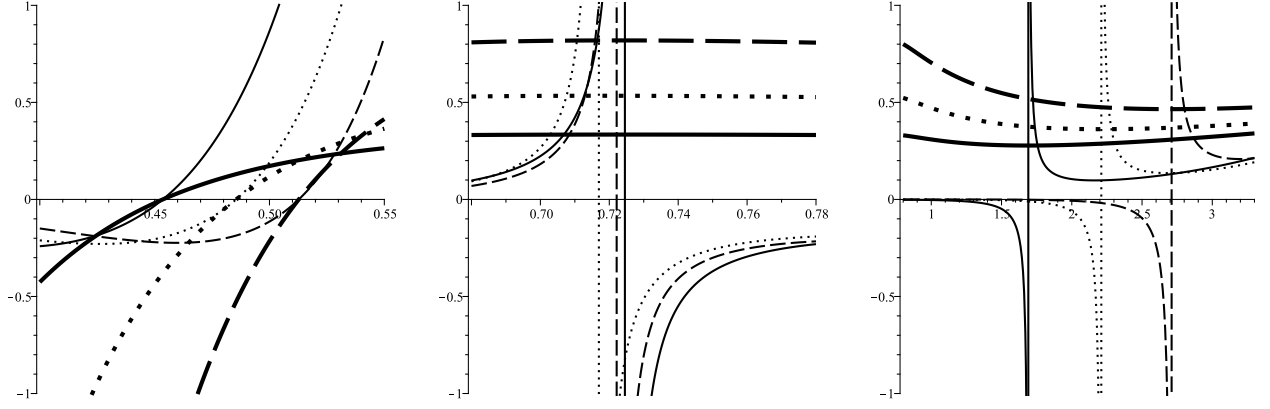


FIG. 10: For different scales ($F(R)$ case): C_Q and T (bold lines) versus r_+ for $q = 1$, $\Lambda = -1$, $g(E) = f(E) = 0.7$ and $f_R = 1.5$; left panel: $d = 6$ (continues line), $d = 7$ (dotted line) and $d = 8$ (dashed line).

$$2g(E)^2 [r\psi'(r) + (d-3)(\psi(r) - 1)] F_R = -r^2 F(R). \quad (40)$$

It is notable that, there are different models of $F(R)$ gravity which may explain some of local (or global) properties of the universe. One of these models which supposed to explain the positive acceleration of expanding universe was $F(R) = R - \frac{\mu}{R}$ model [63]. Another model of $F(R)$ gravity which is introduced by Kobayashi and Maeda [126] is $F(R) = R + \kappa R^n$ (where $n > 1$). This model can resolve the singularity problem arising in the strong gravity regime. On the other hand, by adding an exponential correction term to the Einstein Lagrangian [40, 127, 128], one can prove that this model can satisfy both Solar system tests and high curvature condition [129]. Also, one of the interesting models that passes all the theoretical and observational constraints is known as the Starobinsky model (its functional form is $F(R) = R + \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right)$, [130, 131]). It is notable that, this model could produced viable cosmology different from the Λ CDM one at recent times and also satisfy Solar system and cosmological tests, simultaneously. Moreover, its is worthwhile to mention that there are various models of $F(R)$ gravity in which these models have interesting properties for describing our universe and also Solar system (see [132–148], for an incomplete list of references in this directions).

Besides, another interesting model of $F(R)$ gravity was introduced by Bamba et al. [149]. They constructed an $F(R)$ gravity theory corresponding to the Weyl invariant two scalar field theory ($F(\bar{R}) = \frac{e^{\eta(\varphi(\bar{R}))}}{12} [1 - \varphi(\bar{R})] \bar{R} - e^{2\eta(\varphi(\bar{R}))} J(\varphi(\bar{R}))$, see Ref. [149], for more details). Their model can have the antigravity regions in which the Weyl curvature invariant does not diverge at the Big Crunch and Big Bang singularities. Also, Nojiri and Odintsov investigated the anti-evaporation of Schwarzschild-de Sitter and Reissner-Nordström black holes by several interesting $F(R)$ models such as; $F(R) = \frac{R}{2\kappa^2} + f_0 M^{4-2n} R^n + f_2 R^2$ [150] and $F(R) = \frac{R}{2\kappa^2} \left(1 - \frac{R}{R_0} \right)$ [151].

In order to obtain an exact solution, we should choose a proper model of $F(R)$. In this regard, we use the following interesting models of $F(R)$ gravity with appropriate properties

$$\begin{aligned} F(R) &= R - \lambda e^{-\xi R} + \eta R^n, & \text{type - I} \\ F(R) &= R + \alpha R^n - \beta R^{2-n}, & \text{type - II} \\ F(R) &= R - m^2 \frac{c_1 \left(\frac{R}{m^2} \right)^n}{c_2 \left(\frac{R}{m^2} \right)^n + 1}, & \text{type - III} \end{aligned} \quad (41)$$

$$F(R) = R - a [e^{-bR} - 1] + cR^N \frac{e^{bR} - 1}{e^{bR} + e^{bR_0}}, \quad \text{type - IV}$$

Type-I: This model includes an additional exponential term ($\lambda e^{-\xi R}$). It was shown that such model enjoys validity of the Solar system tests and high curvature condition [129] whereas it suffers instability for its solutions. In order to remove such instability, it is possible to add a correction term (ηR^n with $n > 1$) such as one introduced by Kobayashi and Maeda [126], in which the singularity problem in strong gravity regime is removed. It is worthwhile to mention that for $n = 2$, this model can be used to explain inflation mechanism [127, 128, 130]. The type-I model and some of its properties have been investigated in [51, 68].

Type-II: Considering a scalar potential with non-zero value of residual vacuum energy, another model of $F(R)$ gravity was proposed by Artymowska and Lalak [152]. The reason for such consideration was to have a source for dark energy. This model is a stable minimum of the Einstein frame scalar potential of the auxiliary field. The results of this model are consistent with PLANCK data and also lack of eternal inflation. It should be pointed out that here, α and β are positive constants and n is within the range $[\frac{1}{2}(1 + \sqrt{3}), 2]$ (see Refs. [43, 152], for more details).

Type-III: The third model was introduced by Hu and Sawicki [134]. This model provides a description regarding accelerated expansion without cosmological constant. In addition, this model enjoys the validity of both cosmological and Solar-system tests in the small-field limit. It should be pointed out that, n in this model is positive valued ($n > 0$) where for the case $n = 1$, it covers the mCDTT with a cosmological constant at high curvature regimes. Furthermore, for $n = 2$, the inverse curvature squared model is included [153] with a cosmological constant. In this model, c_1 and c_2 are dimensionless parameters and m^2 is the mass scale (for more details, see Ref. [134]).

Type-IV: The last model is able to describe both inflation in the early universe and the recent accelerated expansion [40]. In addition, this model pass all local tests such as stability of spherical body solutions, generation of a very heavy positive mass for the additional scalar degree of freedom and non-violation of Newton's law. In this model, $N > 2$ and c is a positive quantity (see Ref. [40], for more details).

To find the metric function $\psi(r)$, we use the components of Eqs. (39) and (40) with the models under consideration. Using Eqs. (39) and (40) with the metric (2), one can obtain a general solution for these pure gravity models in the following form

$$\psi(r) = 1 - \frac{2\Lambda r^2}{(d-1)(d-2)g(E)^2} - \frac{m}{r^{d-3}} + \frac{2^{\frac{d-4}{2}} [qf(E)g(E)]^{\frac{d}{2}}}{g(E)^2 r^{d-2}}. \quad (42)$$

In order to satisfy all components of the field equations (Eqs. (39) and (40)), we should set the parameters of $F(R)$ model such that the following equations are satisfied

$$\begin{aligned} \text{type - I: } & \begin{cases} R[(n - \frac{d}{2})\eta R^{n-1} - \frac{d-2}{2}]e^{\xi R} + \lambda(\frac{d}{2} + \xi R) = 0, \\ (1 + n\eta R^{n-1})e^{\xi R} + \lambda\xi = 0, \end{cases} \\ \text{type - II: } & \begin{cases} [\alpha(n - \frac{d}{2})R^n - \frac{d-2}{2}R]R^n + (n + \frac{d-4}{2})\beta R^2 = 0, \\ (\alpha n R^n + R)R^n + (n-2)\beta R^2 = 0, \end{cases} \\ \text{type - III: } & \begin{cases} c_2[\frac{d}{d-2}m^2c_1 - c_2R](\frac{R}{m^2})^{2n} - [\frac{2m^2c_1}{d-2}(n - \frac{d}{2}) + 2c_2R](\frac{R}{m^2})^n - R = 0, \\ [c_2^2(\frac{R}{m^2})^{2n} + 1]R + [2c_2R - m^2nc_1](\frac{R}{m^2})^n = 0, \end{cases} \\ \text{type - IV: } & \begin{cases} [c(N - \frac{d}{2})R^N - \frac{d-2}{2}R - \frac{d}{2}a]e^{3bR} + a(bR + \frac{d}{2})e^{2bR_0} \\ + \{c[(bR + N - \frac{d}{2})e^{bR_0} + bR - N + \frac{d}{2}]R^N \\ - [(d-2)R + ad]e^{bR_0} + a(bR + \frac{d}{2})\}e^{2bR} + \\ [a(2bR + d) - c(N - \frac{d}{2})R^N - (\frac{ad}{2} - \frac{(d-2)R}{2})e^{bR_0}]e^{b(R+R_0)} = 0, \\ (1 + cNR^{N-1})e^{3bR} + [e^{bR_0}(2ab + e^{bR_0}) - cNR^{N-1}e^{bR_0}]e^{bR} \\ + [cR^{N-1}(e^{bR_0}(N + bR) - N + bR) + ab + 2e^{bR_0}]e^{2bR} = 0. \end{cases} \end{aligned} \quad (43)$$

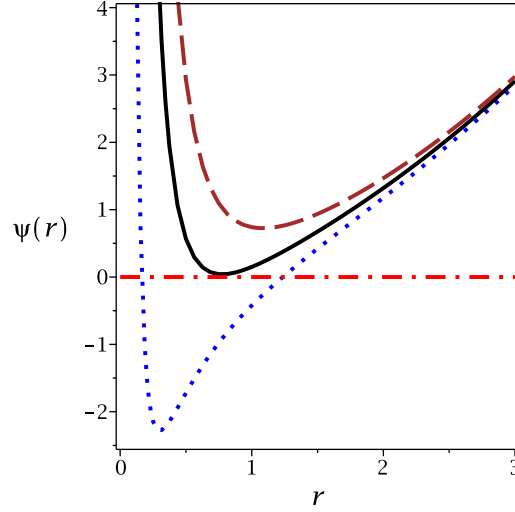


FIG. 11: $\psi(r)$ versus r for $m = 2$, $d = 4$, $f(E) = 1.1$, $\Lambda = -1$, $g(E) = 1.1$, $q = 0.50$ (dotted line), $q = 0.85$ (continuous line) and $q = 1.10$ (dashed line).

Solving Eq. (43), we obtain the parameters of $F(R)$ models as

$$\begin{aligned}
 \text{type - I: } & \begin{cases} \lambda = \frac{R(n-1)e^{\xi R}}{n+\xi R}, \\ \eta = -\frac{(1+\xi R)}{R^{n-1}(n+\xi R)}, \end{cases} \\
 \text{type - II: } & \begin{cases} \alpha = -\frac{1}{2R^{n-1}}, \\ \beta = \frac{R^{n-1}}{2}, \end{cases} \\
 \text{type - III: } & \begin{cases} m^2 = R \left(\frac{n-1}{c_2} \right)^{\frac{-1}{n}}, \\ c_1 = n \left(\frac{n-1}{c_2} \right)^{\frac{1}{n}-1}, \end{cases} \\
 \text{type - IV: } & \begin{cases} c = \frac{[e^{bR} + e^{bR_0}]^2 (bR + 1 - e^{bR})}{R^{N-1} [N(e^{bR} + e^{bR_0}) + bR] (e^{bR} - 1)}, \\ a = -\frac{R \{ (N-1)e^{2bR} + [bR(1+e^{bR_0}) + (N-1)(e^{bR_0} - 1)] e^{bR} \}}{(e^{bR} - 1)^2 [bR + N(e^{bR} + e^{bR_0})]}, \end{cases}
 \end{aligned} \tag{44}$$

Equation (42) is similar to the solutions obtained for Einstein-CIM-gravity's rainbow (Eq. (9)). Calculations show that the Ricci scalar is $R = \frac{2d}{d-2}\Lambda$ and the Kretschmann scalar has the following behaviors

$$\begin{aligned}
 \lim_{r \rightarrow 0} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} & \longrightarrow \infty, \\
 \lim_{r \rightarrow \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} & \longrightarrow \frac{8d}{(d-1)(d-2)^2} \Lambda^2,
 \end{aligned} \tag{45}$$

which confirm that, there is a curvature singularity at $r = 0$. In addition, we find that the asymptotical behavior of the mentioned spacetime is similar to (a)dS Einstein-CIM-rainbow black holes. In other words, one can extract charged solutions from the pure gravity if the $F(R)$ model and its parameters are chosen suitably.

For more investigation of these solutions, we plot $\psi(r)$ versus r . Fig. 11 shows that, the mentioned singularity can be covered with two horizons (Cauchy horizon and event horizon) or one horizon (extreme black hole), and otherwise we encounter with a naked singularity.

B. Thermodynamics

In order to check the first law of thermodynamics for the obtained charged black hole, first, we investigate the Hawking temperature of the pure $F(R)$ gravity's rainbow black holes on the outer horizon r_+ as

$$T_+ = \frac{r_+^{d-2} \left[(d-3) g^2(E) - \frac{2\Lambda r_+^2}{d-2} \right] - 2^{(d-4)/4} [gf(E)g(E)]^{d/2}}{4\pi f(E)g(E)r_+^{d-1}}. \quad (46)$$

Using the method introduced in Ref. [125] for the obtained solutions of pure $F(R)$ gravity's rainbow with $F(R) = F_R = 0$, one may get zero entropy ($S = \frac{A}{4}F_R = 0$, A is the horizon area). Therefore, in order to resolve this problem and for obtain correct nonzero entropy, one may consider the first law of thermodynamics as a fundamental principle. In other words, we can use of $dS = \frac{1}{T}dM$. Using $\partial/\partial t$ as a Killing vector, one can extract the following relation for the finite mass as

$$M = \frac{(d-2)m}{16\pi f(E)g(E)^{d-3}} \quad (47)$$

$$= \frac{\left[(d-2) g^2(E) - \frac{2\Lambda r_+^2}{(d-1)} \right] r_+^{d-2} + 2^{(d-4)/4} (d-2) [gf(E)g(E)]^{d/2}}{16\pi f(E)g(E)^{d-1}r_+}. \quad (48)$$

Hence, we can write

$$dM = \frac{r_+^{d-3} \left[((d-2)(d-3)g^2(E) - 2\Lambda r_+^2) r_+^{d-2} - 2^{(d-4)/4} (d-2) (gf(E)g(E))^{d/2} \right]}{16\pi f(E)g^{d-1}(E)r_+^{d-1}} dr_+, \quad (49)$$

so, we can obtain the entropy in the following form

$$S = \int \frac{1}{T} dM = \frac{1}{4} \left(\frac{r_+}{g(E)} \right)^{d-2}, \quad (50)$$

where it is the area law for the entropy. In other words, considering the $F(R)$ gravity's rainbow with $F_R = 0$, one may use the so-called area law instead of modified area law (Eq. (31)) [125].

C. DK Stability

Here, we are going to discuss about the Dolgov-Kawasaki (DK) stability of these solutions. DK stability in $F(R)$ gravity has been investigated in literature [61, 125, 135, 154–156]. It has been shown that there is no stable ground state for models of $F(R)$ gravity when $F(R) = 0$ and also $F_R = dF(R)/dR \neq 0$ [157]. It is notable that, we find that $F(R) = F_R = 0$, for the obtained solutions (in order to obtain the charged black hole solution in $F(R)$ gravity with constant Ricci scalar, Nojiri and Odintsov in Ref. [151] showed that, $F(R)$ and F_R must be zero). On the other hand, it has been shown that $F_{RR} = d^2F(R)/dR^2$ is related to the effective mass of the dynamical field of the Ricci scalar (see Refs. [158, 159], for more details). So, the positive effective mass is a requirement usually referred to DK stability criterion which leads to stable dynamical field [160, 161]. In order to check this stability, we calculate the second derivative of the $F(R)$ functions with respect to the Ricci scalar for specific models in the following forms

$$F_{RR} = \begin{cases} -\frac{(n-1)[\xi R(n+\xi R)+n]}{R(n+\xi R)}, & \text{type - I} \\ \frac{-(n-1)^2}{R}, & \text{type - II} \\ \frac{n-1}{R}, & \text{type - III} , \\ -\frac{\frac{b^2 R[(n-1)e^{2bR} + (e^{bR_0}(bR+n-1) + bR-n+1)e^{bR} - (n-1)e^{bR_0}]}{(e^{bR}-1)^2 [ne^{bR} + (n+bR)e^{bR_0}]} - \frac{(bR+1-e^{bR})\{n(1-n)b^{3bR} + \tau_1 e^{2bR} - \tau_2 e^{b(R+R_0)} + n(n-1)e^{2bR_0}\}}{(e^{bR}+e^{bR_0})(e^{bR}-1)^2 [ne^{bR} + (n+bR)e^{bR_0}]} R}{(e^{bR}+e^{bR_0})(e^{bR}-1)^2 [ne^{bR} + (n+bR)e^{bR_0}]} R, & \text{type - IV} \end{cases} \quad (51)$$

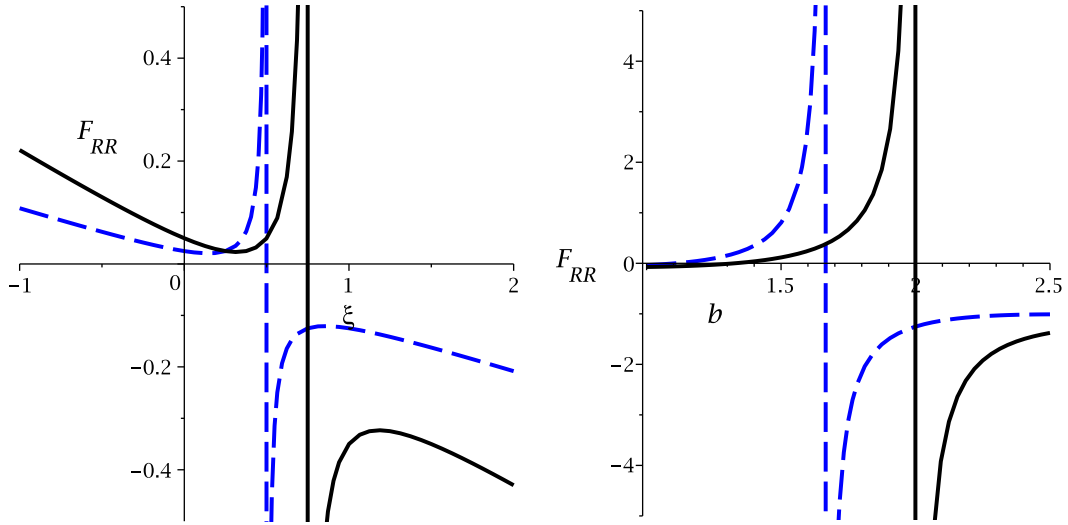


FIG. 12: type-*I*, left panel: F_{RR} versus ξ for $\Lambda = -1$ and $n = 2$ (dashed line), $n = 3$ (continues line). type-*IV*, right panel: F_{RR} versus b for $N = 4$, $R_0 = 4$, $\Lambda = -0.6$ (dashed line), $\Lambda = -0.5$ (continues line).

where

$$\tau_1 = [2n(1-n) + bR(bR-2n)]e^{bR_0} + n(1-n) + bR(bR-2n), \quad (52)$$

$$\tau_2 = [n(n-1) + bR(bR+2n)]e^{bR_0} + 2n(1-n) + bR(bR+2n). \quad (53)$$

In order to have DK stability, F_{RR} must be positive ($F_{RR} > 0$). As one can see, for type-*II* and type-*III* models, the sign of F_{RR} depends on the sign of Ricci scalar. In other words, by considering $\Lambda < 0$ and $\Lambda > 0$, the obtained solutions have DK stability for type-*II* and type-*III*, respectively. On the other hand, for type-*I* and type-*IV*, the sign of F_{RR} is not clear for arbitrary values of parameters. So, we plot F_{RR} in Fig. 12, to analyze the sign of F_{RR} . These figures show that one may obtain stable solutions for special values of parameters of models. In other words, we can set free parameters to obtain stable models.

VI. CONCLUSIONS

In this paper, we have considered higher dimensional Einstein-PMI and Einstein-CIM theories in the presence of an energy dependent spacetime. We obtained metric functions and discussed geometrical properties as well as thermodynamical quantities. We showed that despite the contributions of the gravity's rainbow in thermodynamical quantities, the first law of black holes thermodynamics was valid. Next, we studied pure $F(R)$ gravity as well as $F(R)$ gravity with CIM field in the presence of gravity's rainbow. We pointed out that in case of pure $F(R)$ gravity's rainbow, for the suitable choices of different parameters, one can obtain the electrical charge and cosmological constant, simultaneously. In other words, we showed that, the pure $F(R)$ gravity's rainbow was equivalent to the Einstein-CIM gravity's rainbow. Also, in case of calculations of thermodynamical quantities for the pure $F(R)$ gravity's rainbow, we pointed out that for these classes of black holes, one should employ area law for obtaining entropy instead of the usual modified approaches in case of $F_R = 0$.

In addition, we conducted a study regarding the thermal stability of the obtained solutions in this paper. We pointed out that variations of energy functions in PMI class of the solutions lead to modifications in stability conditions, phase transition points and thermal structure of the solutions while the effects of such variations in $F(R)$ gravity were translation like. We also observed that PMI parameter, s , and $F(R)$ gravity parameter, f_R , were modifying factors in stability conditions and phase transition points. Regarding the effect of dimensionality, we found that it has a translation like behavior, too. An abnormal behavior was observed for smaller divergency in $F(R)$ model.

Regarding the results of this paper, one may regard extended phase space and $P - V$ criticality of the solutions. In addition, we can investigate different viable models of $F(R)$ gravity to study the possible abnormal behavior in thermal stability. Moreover, it is interesting to use a suitable local transformation with appropriate boundary conditions to obtain the so-called Nariai spacetime [162, 163] and discuss the anti-evaporation process [164–167]. We

leave these issues for future work.

Conflict of Interests:

The authors declare that there is no conflict of interests regarding the publication of this paper.

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